Örnek: $Y_1, \ldots, Y_n$ örneklemi üstel dağılımdan rasgele alınsın. $f_Y(y;\theta) = \frac{1}{\theta} \cdot e^{-\frac{y}{\theta}}$, $y > 0$

EGOT tahmin edicisinin $\hat{\theta} = \overline{Y}$ varyansını Rao-Cramer alt sınırı ile karşılaştırınız.

EGOT,

$$L(y;\theta) = \prod_{i=1}^{n} f(y_i;\theta)$$

$$= \frac{1}{\theta^n} \cdot e^{-\frac{\sum y}{\theta}}$$

$$\ln L(y;\theta) = -n \cdot \ln(\theta) - \frac{\sum y}{\theta}$$

$$\Rightarrow \frac{\partial \ln L(y;\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum y}{\theta^2} = 0$$

$$\Rightarrow -n \cdot \theta + \sum y = 0$$

$$\Rightarrow \hat{\theta} = \frac{\sum y}{n} = \overline{y} \text{ 'dir.}$$

$$E(y) = \int_0^\infty \underbrace{y}_{u} \cdot \frac{1}{\theta} \cdot \underbrace{e^{-\frac{y}{\theta}} \cdot dy}_{dv}$$

$y = u \Rightarrow dy = du$

$e^{-\frac{y}{\theta}} \cdot dy = dv$

$-\theta \cdot e^{-\frac{y}{\theta}} = v$

$$= \frac{1}{\theta} \cdot \left[y \cdot \left(-\theta \cdot e^{-\frac{y}{\theta}}\right) \Big|_0^\infty + \int_0^\infty \theta \cdot e^{-\frac{y}{\theta}} \, dy \right]$$

$$= \frac{1}{\theta} \cdot \left[-\theta^2 \cdot e^{-\frac{y}{\theta}} \Big|_0^\infty \right] = +\theta \cdot \left[0 + e^0\right] = \theta //$$

$$E(y^2) = \int_0^\infty \underbrace{y^2}_{u} \cdot \frac{1}{\theta} \cdot \underbrace{e^{-\frac{y}{\theta}} \cdot dy}_{dv}$$

$y^2 = u \Rightarrow 2y\,dy = du$

$dy = \frac{du}{2y}$

$e^{-\frac{y}{\theta}} \cdot dy = dv$

$-\theta \cdot e^{-\frac{y}{\theta}} = v$

$$= \frac{1}{\theta} \left[y^2 \cdot \left(-\theta \cdot e^{-\frac{y}{\theta}}\right) \Big|_0^\infty - \int_0^\infty -\theta \cdot e^{-\frac{y}{\theta}} \cdot 2y \cdot dy \right]$$

$$= \frac{1}{\theta} \cdot \left[2 \int_0^\infty \theta \cdot y \cdot e^{-\frac{y}{\theta}} \, dy \right]$$

$$= \frac{1}{\theta} \cdot \left[2 \cdot \theta^2 \underbrace{\int y \cdot \frac{1}{\theta} \cdot e^{-\frac{y}{\theta}} dy}_{= E(y) = \theta} \right] = 2\theta^2 //$$

$\Rightarrow V(Y) = 2\theta^2 - \theta^2 = \theta^2$ olur.

$V(\hat{\theta}) = V(\bar{Y}) = V\left(\frac{1}{n}\sum Y\right) = \frac{1}{n^2} \cdot \sum V(Y)$

$= \frac{1}{n^2} \cdot n \cdot \theta^2 = \frac{\theta^2}{n}$ //

Rao-Cramer Alt sınırı ile

$f_Y(y;\theta) = \frac{1}{\theta} \cdot e^{-\frac{y}{\theta}}$

$\ln f_Y(y;\theta) = \ln 1 - \ln\theta - \frac{y}{\theta}\ln(e)$

$\Rightarrow \frac{\partial \ln f_Y(y;\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{y}{\theta^2}$

$\Rightarrow \left[\frac{\partial \ln f(y;\theta)}{\partial\theta}\right]^2 = \left(-\frac{1}{\theta} + \frac{y}{\theta^2}\right)^2 = \frac{1}{\theta^2} - \frac{2y}{\theta^3} + \frac{y^2}{\theta^4}$

$\Rightarrow E\left[\frac{1}{\theta^2} - \frac{2y}{\theta^3} + \frac{y^2}{\theta^4}\right] = \frac{1}{\theta^2} - \frac{2 \cdot E(y)}{\theta^3} + \frac{E(y^2)}{\theta^4}$

$= \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} + \frac{2\theta^2}{\theta^4}$

$= \frac{1}{\theta^2}$ //

$\Rightarrow V(\hat{\theta}) \geq \frac{1}{n \cdot E\left[\frac{\partial \ln f(y;\theta)}{\partial\theta}\right]^2}$

$\Rightarrow \frac{\theta^2}{n} \geq \frac{1}{n \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{n}$

old. dan $\hat{\theta} = \bar{Y}$ tahmini yansız, Min. varyanslı ve etkindir.

Örnek: $Y_1, \ldots, Y_n$; $f_Y(y;\theta) = \frac{2y}{\theta^2}$, $0 < y < \theta$

$\hat{\theta} = \frac{3}{2}\bar{Y}$, θ parametresinin yansız tahmin edicisi ise $V(\hat{\theta})$'nın RC alt sınırından küçük old. göst?

$$E(Y) = \int_0^\theta y \cdot \frac{2y}{\theta^2} \cdot dy = \frac{2}{\theta^2} \cdot \frac{y^3}{3}\Big|_0^\theta = \frac{2\theta^3}{3\theta^2} = \frac{2}{3}\theta$$

$$E(\hat{\theta}) = E\left(\frac{3}{2}\bar{Y}\right) = \frac{3}{2}\cdot\left(\frac{2}{3}\theta\right) = \theta \text{ , } \hat{\theta} = \frac{3}{2}\bar{Y}, \theta \text{ için yansız.}$$

$$E(Y^2) = \int_0^\theta y^2 \frac{2y}{\theta^2} \cdot dy = \frac{2}{\theta^2} \cdot \frac{y^4}{4}\Big|_0^\theta = \frac{\theta^2}{2} //$$

$$\Rightarrow V(Y) = E(Y^2) - [E(Y)]^2 = \frac{\theta^2}{2} - \left(\frac{2}{3}\theta\right)^2 = \frac{\theta^2}{2} - \frac{4}{9}\theta^2$$

$$= \frac{\theta^2}{18} //$$

$$\Rightarrow V(\hat{\theta}) = V\left(\frac{3}{2}\bar{Y}\right) = \frac{9}{4} \cdot V\left(\frac{\sum Y}{n}\right) = \frac{9}{4n^2} \cdot \sum V(Y)$$

$$= \frac{9}{4n^2} \cdot n \cdot \frac{\theta^2}{18} = \frac{\theta^2}{8n} \text{ olur.}$$

(Linear özellik)

RC alt sınırı ise

$$f_Y(y,\theta) = \frac{2y}{\theta^2}$$

$$\ln f_Y(y,\theta) = \ln 2y - 2\ln\theta$$

$$\Rightarrow \frac{\partial \ln f_Y(y;\theta)}{\partial\theta} = -\frac{2}{\theta}$$

$$\Rightarrow \frac{1}{n \cdot E\left(\frac{\partial \ln f(y;\theta)}{\partial\theta}\right)^2} = \frac{1}{n \cdot E\left(-\frac{2}{\theta}\right)^2} = \frac{1}{n \cdot E\left(\frac{4}{\theta^2}\right)}$$

$$= \frac{1}{\frac{4n}{\theta^2}} = \frac{\theta^2}{4n} //$$

$V(\hat{\theta}) = \frac{\theta^2}{8n} < \frac{\theta^2}{4n}$

Tes. def. nin tanım aralığı θ'ya bağlı old. için RC eşitsizliği varsayımı bozuldu. RC uygulanamaz.